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A Sampling Theorem with Nonuniform Complex Nodes

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Using contour integration and a multiplier technique, we establish a sampling theorem with nonuniform complex nodes $(t_n)_{n\in\mathbb{Z}}$ which applies to entire functions of exponential type including band-limited L^2-functions. The sequence $(t_n)_{n\in\mathbb{Z}}$ must satisfy $\sup_{n\in\mathbb{Z}} |\Re(t_n)-n| < \infty$ and $\sup_{n\in\mathbb{Z}} |\Im(t_n)| < \infty$. The sampled function may grow faster than any polynomial on the real line. © 1997 Academic Press

1. INTRODUCTION AND STATEMENT OF RESULTS

In recent years several authors [2, 8–10, 14] have established various sampling theorems with nonuniform real nodes by using the method of contour integration. There are also sampling theorems with nonuniform complex nodes [7, 17]. However, their proofs are based on Hilbert space methods and consequently they apply to band-limited L^2-functions only.

In this paper we shall extend the method of contour integration to the case of nonuniform complex nodes. Our main result is a Lagrange-type interpolation formula (see Theorem 1.1) that applies to a class of entire functions of exponential type which is considerably wider than the class of band-limited L^2-functions. The admissible functions may even grow faster than any polynomial on the real line (see Corollary 1.2). As a consequence, we also obtain a uniqueness theorem for entire functions of exponential type which is much more general than the classical results [1, Chap. 9] as far as freedom of the nodes is concerned (see Corollary 1.3).

As usual, let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural, integer, real, and complex numbers. For a complex number z we denote its real and imaginary part by $\Re(z)$ and $\Im(z)$. Throughout this paper the nodes $(t_n)_{n\in\mathbb{Z}}$ are subject to the following conditions:

There exist positive integers L and N with $N > L$ and positive real numbers δ and I such that

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$$t_n \neq 0 \qquad \text{for} \quad n \neq 0; \tag{1}$$

$$|\Re(t_n) - n| \leqslant L \qquad \text{for} \quad |n| \geqslant N; \tag{2}$$

$$\Re(t_{n+1}) - \Re(t_n) > \delta \qquad \text{for all integers } n; \tag{3}$$

$$|\Im(t_n)| \leqslant I \qquad \text{for} \quad |n| \geqslant N; \tag{4}$$

$$|t_n| \leqslant |n| + L \qquad \text{for} \quad |n| \geqslant N. \tag{5}$$

First a few comments on these properties. Conditions (1)–(3) are the standard hypotheses in sampling with nonuniform real nodes and are of relevance in growth theorems such as the theorem of Duffin and Schaeffer [1, p. 191]. Condition (3) ensures that the sequence $(\Re(t_n))_{n\in\mathbb{Z}}$ is strictly increasing and separated. If we restrict ourselves to real nodes, then (4) is trivially satisfied and (5) is a consequence of (2). Thus in this case, our conditions reduce to the standard ones. Note that (2) and (3) imply that δ is 1 at most.

Now we define the canonical product G corresponding to $(t_n)_{t\in\mathbb{Z}}$ by

$$G(z) := (z - t_0) \prod_{n=1}^{\infty} \left(1 - \frac{z}{t_n}\right)\left(1 - \frac{z}{t_{-n}}\right). \tag{6}$$

Since

$$\left(1 - \frac{z}{t_n}\right)\left(1 - \frac{z}{t_{-n}}\right) = 1 + \frac{z^2 - z(t_n + t_{-n})}{t_n t_{-n}}$$

and

$$\left|\frac{z^2 - z(t_n + t_{-n})}{t_n t_{-n}}\right| \leqslant \frac{|z|^2 + |z|\,(2L + 2I)}{(n - L)^2}$$

for all integers n with $|n| \geqslant N$, the product G converges absolutely and uniformly on all compact subsets of $\mathbb{C}$ and therefore represents an entire function.

We give two examples of a function G given by (6).

Example 1. Since the zeros $(j_n)_{n\in\mathbb{Z}}$ of the function $J_\nu(z)/z^\nu$, where J_ν is the Bessel function of order ν, satisfy $j_n = n\pi + c + O(1/n)$ as $n \to \infty$ and $j_{-n} = -j_n$ for $n \in \mathbb{N}$, the nodes $(t_n)_{n\in\mathbb{Z}}$ defined by $t_n := j_n/\pi$ fulfill (1)–(5). The canonical product G corresponding to the sequence $(t_n)_{n\in\mathbb{Z}}$ is given by $J_\nu(\pi z)\Gamma(\nu+1)2^\nu/(\pi z)^\nu$ (cf. [16]). Kramer [11] proved a sampling theorem for the nodes $(j_n)_{n\in\mathbb{Z}}$. It is known that there is a connection between Kramer's sampling theorem and sampling expansions generated by Lagrange interpolation (e.g., [18]).

Example 2. Obviously, the sequence $(t_n)_{n \in \mathbb{Z}}$ defined by $t_n = n + t$ for some fixed complex number t and all integers n satisfies the conditions (1)–(5). A simple calculation yields that the canonical product G corresponding to $(t_n)_{n \in \mathbb{Z}}$ is given by

$$G(z) = \frac{it}{\sinh i\pi t} \sin(\pi(z-t)).$$

If t is equal to zero then G reduces to $(1/\pi) \sin \pi z$. In general, the canonical product G is not obtainable in closed form.

Our result is as follows.

THEOREM 1.1. *Let $(t_n)_{n \in \mathbb{Z}}$ be a sequence of nodes satisfying* (1)–(5). *Let f and Φ be entire functions of exponential types σ and ε such that*

$$\sigma + \varepsilon \leqslant \pi \tag{7}$$

and

$$|f(x)\, \Phi(x-\zeta)| \leqslant C_1(\zeta)(|x|+1)^{-4L} \qquad \text{for} \quad x \in \mathbb{R}, \quad \zeta \in \mathbb{C}, \tag{8}$$

where $C_1(\cdot)$ is positive and bounded on compact subsets of $\mathbb{C}$.

Then

$$f(z)\, \Phi(0) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{\Phi(t_n - z)}{z - t_n} \frac{G(z)}{G'(t_n)} \tag{9}$$

for all $z \in \mathbb{C}$. Moreover, the convergence of the series is uniform on every compact subset of $\mathbb{C}$.

To get a sampling theorem for a large class of entire functions, it is obviously desirable to choose a function Φ whose modulus on the real line tends to zero rapidly.

A suitable example for the function Φ is given by

$$\Phi(z) := \Phi_{\varepsilon, k}(z) := \left(\frac{\sin(\varepsilon z/k)}{\varepsilon z/k} \right)^k, \tag{10}$$

where ε is a positive real number and k a positive integer. A simple consideration shows that $\Phi_{\varepsilon, k}$ is of exponential type ε and satisfies

$$|\Phi_{\varepsilon, k}(x)| = O(|x|^{-k}) \qquad \text{as} \quad x \to \pm\infty.$$

Therefore, choosing Φ as in (10) with $0 < \varepsilon < \pi$ and $k \in \mathbb{N}$, we can apply Theorem 1.1 to all entire functions f of exponential type $\pi - \varepsilon$ satisfying

$$|f(x)| = O(|x|^{k-4L}) \qquad \text{as} \quad x \to \pm\infty.$$

Let us mention that the multiplier Φ given by (10) has been used by various authors for the same purpose (see, e.g., [9, p. 81; 14; 15]).

There are also other possibilities of a suitable choice of Φ. Given $\alpha > 1$, $\varepsilon > 0$, an entire function $\psi(\alpha, \varepsilon, \cdot)$ of exponential type ε has been constructed in [4], a function which is nearly the best possible choice. More precisely, its growth on the real line is given by

$$|\psi(\alpha, \varepsilon, x)| = O\left(\exp\left(-\frac{|x|}{(\log |x|)^{\alpha}}\right)\right) \qquad \text{as} \quad x \to \pm\infty.$$

Note that if φ is a non-trivial entire function of exponential type satisfying

$$|\varphi(x)| = O(\exp(-w(|x|))) \qquad \text{as} \quad x \to \pm\infty,$$

where $w(\cdot)$ is positive, then necessarily (cf. [4])

$$\int_1^{\infty} \frac{w(x)}{x^2}\, dx < \infty.$$

We may assume that $\psi(\alpha, \varepsilon, 0) = 1$. Otherwise, we can consider the function $\tilde{\psi}$ given by

$$\tilde{\psi}(z) := \frac{k!}{\psi^{(k)}(\alpha, \varepsilon, 0)} \frac{\psi(\alpha, \varepsilon, z)}{z^k},$$

where k is the order of $\psi(\alpha, \varepsilon, \cdot)$ at zero. The function $\tilde{\psi}$ is also entire and of exponential type ε, has the same asymptotic behavior as $\psi(\alpha, \varepsilon, \cdot)$, and satisfies $\tilde{\psi}(0) = 1$.

Although the authors [4] gave a construction of the function $\psi(\alpha, \varepsilon, \cdot)$, it is not easily available for numerical purposes. However, in the following application it is enough to know the existence of $\psi(\alpha, \varepsilon, \cdot)$.

With ψ taking the role of Φ in Theorem 1.1, we obtain the following result which extends a theorem of Rahman and Schmeisser [12, Theorem 3] from equidistant to nonuniform complex nodes.

Corollary 1.2. *Let $(t_n)_{n \in \mathbb{Z}}$ be a sequence of nodes satisfying* (1)–(5). *Let f be an entire function of exponential type $\sigma < \pi$ satisfying*

$$|f(x)| = O\left(\exp\left(\frac{|x|}{(\log |x|)^{\lambda}}\right)\right) \qquad \textit{as} \quad x \to \pm\infty, \tag{11}$$

where $\lambda > 1$.

Then, choosing $\varepsilon \in (0, \pi - \sigma]$, $\alpha \in (1, \lambda)$, *and* $\Phi := \psi((1+\alpha)/2, \varepsilon, \cdot)$, *the following equality holds,*

$$f(z) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{\Phi(t_n - z)}{z - t_n} \frac{G(z)}{G'(t_n)} \tag{12}$$

for all $z \in \mathbb{C}$, *where the series converges uniformly on all compact subsets of* $\mathbb{C}$.

Note that in the corollary the conditions for f are independent of the numbers L and I which control the deviation of t_n from n ($n \in \mathbb{Z}$).

As an immediate consequence of Corollary 1.2, we obtain the following

COROLLARY 1.3. *Let* f *be an entire function of exponential type* $\sigma < \pi$ *satisfying*

$$|f(x)| = O\left(\exp\left(\frac{|x|}{(\log |x|)^{\lambda}}\right)\right) \qquad as \quad x \to \pm\infty,$$

where $\lambda > 1$. *If* f *vanishes on a sequence* $(t_n)_{n \in \mathbb{Z}}$ *of points subject to the conditions* (1)–(5), *then* f *is identically zero.*

2. LEMMAS

Our assumptions on the nodes imply that $\Re(t_n) > 0$ and $\Re(t_{-n}) < 0$ for $n \geqslant N$. Let $\eta := \delta/4$, which is 1/4 at most since $\delta \leqslant 1$ (see above). Then, as a consequence of (3), we are able to construct two sequences of positive real numbers $(R_m^+)_{m \geqslant N}$ and $(R_m^-)_{m \geqslant N}$ with the following properties:

$$\left.\begin{array}{l} \Re(t_m) + \eta < R_m^+ < \Re(t_{m+1}) - \eta \\ \Re(t_{-m}) - \eta > -R_m^- > \Re(t_{-(m+1)}) + \eta \end{array}\right\} \quad \text{for all} \quad m \geqslant N, \tag{13}$$

$$\left.\begin{array}{l} |R_m^+ - n| > \eta \\ |R_m^- - n| > \eta \end{array}\right\} \quad \text{for all} \quad m \geqslant N \quad \text{and} \quad n \in \mathbb{N}. \tag{14}$$

By a simple calculation we obtain the following

LEMMA 2.1. *Under the hypotheses* (1)–(5) *and* (13) *and* (14) *there exists an integer* $S \geqslant N$ *so that for all* $m \geqslant S$, $\varphi \in [-\pi/2, \pi/2]$, *and* $n \geqslant N$ *we have*

$$\left.\begin{array}{l} |R_m^+ e^{i\varphi} - t_n| \\ |-R_m^- e^{i\varphi} - t_{-n}| \end{array}\right\} > \frac{\eta}{2}. \tag{15}$$

Proof. We shall prove only the first inequality. The proof of the second inequality is very similar.

The conditions (3) and (4) reveal that for $N \leqslant n \leqslant m$ the point t_n lies inside the circle of radius $|\Re(t_m) + iI|$ centered at the origin, whereas for $N \leqslant m < n$ it lies outside the concentric circle of radius $\Re(t_{m+1})$. An elementary calculation shows that

$$R_m^+ - |\Re(t_m) + iI| > \frac{\eta}{2}$$

if

$$\frac{I^2}{\Re(t_m)} < \eta.$$

Clearly, this condition is satisfied for sufficiently large m. Hence there exists an integer $S \geqslant N$ such that

$$|R_m^+ e^{i\varphi} - t_n| \geqslant \min\{R_m^+ - |\Re(t_m) + iI|, \Re(t_{m+1}) - R_m^+\} > \frac{\eta}{2}$$

for all $n \geqslant N$ and $m \geqslant S$. ∎

In the following we shall always represent the nodes as

$$t_n =: r_n e^{i\theta_n} \qquad \text{with} \quad r_n \in \mathbb{R} \quad \text{and} \quad \theta_n \in [-\pi/2, \pi/2] \tag{16}$$

($n \in \mathbb{Z}$). Note that by this convention r_n is not restricted in sign. More precisely, r_n and n are of the same sign provided that $|n| \geqslant N$.

LEMMA 2.2. *Let $K \in \mathbb{N}$ and $j \in \mathbb{Z}$ with $K + j \geqslant N$. Then the infinite product*

$$P(m, \varphi) := \prod_{n=K}^{\infty} \left| \frac{n + R_m^+ e^{i(\varphi - \theta_{-(n+j)})}}{n + R_m^+ e^{i\varphi}} \right|$$

converges absolutely for all $m \geqslant N$ and $\varphi \in [-\pi/2, \pi/2]$. Furthermore, there exist a positive real number C_2 and an integer $S \geqslant N$ such that

$$P(m, \varphi) \geqslant C_2 \tag{17}$$

for all $m \geqslant S$ and $\varphi \in [-\pi/2, \pi/2]$.

Proof. Without loss of generality we may assume that $\varphi \in [0, \pi/2]$. Otherwise, we can argue with the sequence $(\bar{t}_n)_{n \in \mathbb{Z}}$, which also satisfies the hypotheses (1)–(5). Defining

$$F(n, m, \varphi) := \left| \frac{n + R_m^+ e^{i(\varphi - \theta_{-(n+j)})}}{n + R_m^+ e^{i\varphi}} \right|,$$

we obtain by a straightforward calculation that

$$(F(n, m, \varphi))^2 = 1 + \frac{2nR_m^+(\cos(\varphi - \theta_{-(n+j)}) - \cos\varphi)}{|n + R_m^+ e^{i\varphi}|^2}$$

$$= 1 + \frac{4nR_m^+}{|n + R_m^+ e^{i\varphi}|^2} \sin\left(\frac{\theta_{-(n+j)}}{2}\right) \sin\left(\frac{2\varphi - \theta_{-(n+j)}}{2}\right).$$

As a consequence of (2) and (4), we find for the modulus of

$$g(n, m, \varphi) := (F(n, m, \varphi))^2 - 1$$

that

$$|g(n, m, \varphi)| \leqslant \frac{4nR_m^+}{|n + R_m^+ e^{i\varphi}|^2} \frac{I}{n + j - L} \leqslant \frac{4C_3 R_m^+}{|n + R_m^+ e^{i\varphi}|^2},$$

where $C_3 := \sup\{nI/(n+j-L) : n \geqslant K\} < \infty$.

Since $|n + R_m^+ e^{i\varphi}|^2 \geqslant n^2$, the infinite product $\prod_{n=K}^{\infty} (F(n, m, \varphi))^2$ converges absolutely. Using the inequality $|\sqrt{x} - 1| \leqslant |x - 1|$, which holds for positive x, we deduce that $P(m, \varphi)$ also converges absolutely.

Let us choose $S \geqslant N$ so that for all $m \geqslant S$, $n \geqslant K$, and $\varphi \in [0, \pi/2]$ we have

$$\frac{4C_3 R_m^+}{|n + R_m^+ e^{i\varphi}|^2} \leqslant \frac{1}{2}.$$

Now applying the inequality $e^{-2|x|} \leqslant 1 + x$, which holds for $x \in [-\frac{1}{2}, \infty)$, we find for all $m \geqslant S$ that

$$(P(m, \varphi))^2 \geqslant \prod_{n=K}^{\infty} \exp(-2\,|g(n, m, \varphi)|) \geqslant \exp\left(-2 \sum_{n=K}^{\infty} \frac{4C_3 R_m^+}{|n + R_m^+ e^{i\varphi}|^2}\right).$$

Hence

$$P(m, \varphi) \geqslant \exp\left(-4C_3 R_m^+ \sum_{n=K}^{\infty} \frac{1}{n^2 + (R_m^+)^2}\right)$$

$$\geqslant \exp\left(-4C_3 R_m^+ \int_{K-1}^{\infty} \frac{dx}{x^2 + (R_m^+)^2}\right)$$

$$= \exp\left(-4C_3 \left(\frac{\pi}{2} - \arctan\frac{K-1}{R_m^+}\right)\right)$$

$$\geqslant \exp(-2\pi C_3),$$

which shows that (17) holds, too. ∎

A useful result is the following

LEMMA 2.3. *Let J be a non-negative real number. Then*

$$|\sin(\pi(z-iJ))| = \frac{\sinh(\pi J)}{J}\,|z-iJ| \prod_{n=1}^{\infty} \left|1-\frac{z}{n+iJ}\right| \left|1-\frac{z}{-n+iJ}\right|$$

holds for all complex numbers z.

Proof. Using the representations of sin and sinh by infinite products [6, p. 44, Sect. 1.431], we obtain that

$$\frac{|\sin(\pi(z-iJ))|}{\sinh(\pi J)} = \frac{\pi\,|z-iJ| \prod_{n=1}^{\infty} \left|1-\frac{z-iJ}{n}\right| \left|1-\frac{z-iJ}{-n}\right|}{\pi J \prod_{n=1}^{\infty} \left|1+\frac{iJ}{n}\right| \left|1+\frac{iJ}{-n}\right|}$$

$$= \frac{|z-iJ|}{J} \prod_{n=1}^{\infty} \left|1-\frac{z}{n+iJ}\right| \left|1-\frac{z}{-n+iJ}\right|. \quad \blacksquare$$

Now we are able to find an estimate for the growth of the canonical product defined in (6).

LEMMA 2.4. *Let $(t_n)_{n\in\mathbb{Z}}$ be a sequence of nodes satisfying* (1)–(5). *Let G be the canonical product corresponding to $(t_n)_{n\in\mathbb{Z}}$ and let the sequences $(R_m^+)_{m\geqslant N}$ and $(R_m^-)_{m\geqslant N}$ be subject to* (13) *and* (14).
Then there exists an integer $S \geqslant N$ so that for all $m \geqslant S$ we have

$$|G(R_m^+ e^{i\varphi})| \geqslant C_4 (R_m^+)^{-2L}\, H(R_m^+ e^{i\varphi}) \qquad \text{if} \quad \varphi \in (-\pi/2, \pi/2), \tag{18}$$

$$|G(R_m^- e^{i\varphi})| \geqslant C_5 (R_m^-)^{-2L}\, H(R_m^- e^{i\varphi}) \qquad \text{if} \quad \varphi \in (\pi/2, 3\pi/2), \tag{19}$$

where H is defined by

$$H(Re^{i\varphi}) := \begin{cases} R^{-2L} & \text{if} \quad |\sin\varphi| \leqslant (4I+2L)/R \\ e^{\pi(R\,|\sin\varphi| - I)}\, |\sin\varphi|^{2L} & \text{if} \quad |\sin\varphi| > (4I+2L)/R \end{cases}$$

for all positive real numbers R. The positive real numbers C_4 and C_5 are independent of m and φ.

Proof. We may restrict ourselves to a proof of (18) for $\varphi \in [0, \pi/2)$ as can be seen from the following. Along with $(t_n)_{n\in\mathbb{Z}}$ the sequences $(\bar{t}_n)_{n\in\mathbb{Z}}$,

$(-t_{-n})_{n\in\mathbb{Z}}$, and $(-\bar{t}_{-n})_{n\in\mathbb{Z}}$ also satisfy the hypotheses (1)–(5). Hence, if we apply inequality (18) for $\varphi\in[0,\pi/2)$ to the canonical products

$$G_1(z)=(z-\bar{t}_0)\prod_{n=1}^{\infty}\left(1-\frac{z}{\bar{t}_n}\right)\left(1-\frac{z}{\bar{t}_{-n}}\right),$$

$$G_2(z)=(z-(-t_0))\prod_{n=1}^{\infty}\left(1-\frac{z}{-t_{-n}}\right)\left(1-\frac{z}{-t_n}\right),$$

$$G_3(z)=(z-(-\bar{t}_0))\prod_{n=1}^{\infty}\left(1-\frac{z}{-\bar{t}_{-n}}\right)\left(1-\frac{z}{-\bar{t}_n}\right),$$

we arrive at (18) for $\varphi\in(-\pi/2,0]$ and (19) for $\varphi\in[\pi,3\pi/2)$ and $\varphi\in(\pi/2,\pi]$.

For $\varphi\in[0,\pi/2)$ and $m\geqslant N$ we introduce

$$z_m:=x_m+iy_m:=R_m^+e^{i\varphi}.$$

Clearly, x_m, y_m, and z_m depend on φ. For convenience we do not express this fact in our notation but keep it in mind in the following consideration.

In the following C_j $(j=6,\ldots,15)$ and S_j $(j=1,\ldots,4)$ denote appropriate positive numbers which do not depend on m or φ. We do not need them explicitly but in cases where their value is easily accessible we indicate their construction.

Let us choose a positve integer S_1 satisfying

$$S_1>\max\{|n+iI|,|t_n|:|n|\leqslant N-1\}+L+1.$$

Then the function

$$h(z):=\frac{(z-t_0)\prod_{n=1}^{N-1}\left(1-\frac{z}{t_n}\right)\left(1-\frac{z}{t_{-n}}\right)}{(z-iI)\prod_{n=1}^{N-1}\left(1-\frac{z}{n+iI}\right)\left(1-\frac{z}{-n+iI}\right)}$$

is defined for all $z\in D:=\{\xi\in\mathbb{C}:|\xi|\geqslant S_1-L\}$. Furthermore, there exists a positive real number C_6 such that

$$|h(z)|\geqslant C_6$$

for all $z\in D$. We define

$$P(z):=\prod_{n=N}^{\infty}\left(1-\frac{z}{t_n}\right)\left(1-\frac{z}{t_{-n}}\right).$$

Since $z_m \in D$ for all $m \geqslant S_1$, we have

$$|G(z_m)| \geqslant C_6 \, |z_m - iI| \prod_{n=1}^{N-1} \left|1 - \frac{z_m}{n+iI}\right| \left|1 - \frac{z_m}{-n+iI}\right| |P(z_m)| \tag{20}$$

for all $m \geqslant S_1$. We shall find a lower bound for $|P(z_m)|$ mainly by geometric arguments.

Using (16) and noting that $r_{-n} < 0$ for $n \geqslant N$, we can easily see that

$$|P(z_m)| \geqslant \prod_{n=N}^{\infty} \left|1 - \frac{z_m}{t_n}\right| \left|1 - \frac{z_m e^{i|\theta_{-n}|}}{r_{-n}}\right|.$$

For $\varphi + |\theta_{-n}| \leqslant \pi/2$ it follows from (5) that for all $n \geqslant N$

$$\left|1 - \frac{z_m e^{i|\theta_{-n}|}}{r_{-n}}\right| \geqslant \left|1 + \frac{z_m e^{i|\theta_{-n}|}}{n+L}\right|. \tag{21}$$

A geometrical reflection shows that in the case of $\varphi + |\theta_{-n}| > \pi/2$ the inequality (21) is also valid if

$$\cos(\pi - (\varphi + |\theta_{-n}|)) \leqslant \frac{R_m^+}{n+L}. \tag{22}$$

But (22) is satisfied as soon as $R_m^+ \geqslant I(N+L)/(N-L)$. Indeed, under that restriction

$$\cos(\pi - (\varphi + |\theta_{-n}|)) \leqslant \sin |\theta_{-n}| \leqslant \frac{I}{|t_{-n}|} \leqslant \frac{I}{n-L} \leqslant \frac{R_m^+}{n+L}$$

for $n \geqslant N$. Thus, in conjunction with Lemma 2.2, we find that

$$|P(z_m)| \geqslant C_2 \prod_{n=N}^{\infty} \left|1 - \frac{z_m}{t_n}\right| \left|1 + \frac{z_m}{n+L}\right| \tag{23}$$

for $m \geqslant S_2 := \max\{S, S_1, I(N+L)/(N-L) + L\}$, where S is chosen according to Lemma 2.2. For a lower bound of $|1 - z_m/t_n|$ we distinguish two cases which correspond to those in the definition of the function H of our lemma.

Case 1. Let $\Im(z_m) = y_m > 4I + 2L$.

Since $\varphi \in [0, \pi/2)$, we have $\Re(z_m) = x_m > 0$. Defining $p_n := \Re(t_n)$, we find that

$$\left|1 - \frac{z_m}{t_n}\right|^2 \geqslant \frac{(p_n - x_m)^2 + (y_m - I)^2}{p_n^2 + I^2}.$$

A discussion of the function

$$f(t) := \frac{(t - x_m)^2 + (y_m - I)^2}{t^2 + I^2}$$

by standard methods of calculus shows that f has an absolute minimum at

$$\tau_m := \frac{1}{2x_m}((R_m^+)^2 - 2y_m I + \sqrt{((R_m^+)^2 - 2y_m I)^2 + 4x_m^2 I^2})$$

and is strictly decreasing for $t \in [0, \tau_m]$ and strictly increasing for $t \in [\tau_m, \infty)$. As a consequence, we obtain that

$$\left|1 - \frac{z_m}{t_n}\right| \geqslant \left|1 - \frac{z_m}{n + L + iI}\right| \qquad \text{if} \quad n \leqslant \tau_m - L$$

and

$$\left|1 - \frac{z_m}{t_n}\right| \geqslant \left|1 - \frac{z_m}{n - L + iI}\right| \qquad \text{if} \quad n \geqslant \tau_m + L.$$

For all $m \geqslant S_2$ we find the following estimate for τ_m:

$$\frac{R_m^+ - 2I}{\cos \varphi} \leqslant \tau_m. \tag{24}$$

Therefore, for all $m \geqslant S_3 := \max\{S_2, 2I + 2L + N + 1\}$ we have

$$\lfloor \tau_m \rfloor \geqslant N + L,$$

where $\lfloor x \rfloor$ denotes the integer part of x. Since

$$\left|1 + \frac{z_m}{n}\right| = \left|\frac{n + z_m}{n + z_m - iI}\right| \left|\frac{n - iI}{n}\right| \left|1 - \frac{z_m}{-n + iI}\right| > \left|1 - \frac{z_m}{-n + iI}\right|,$$

it follows from (23) for all $m \geqslant S_3$ that

$$|P(z_m)| \geqslant C_2 \prod_{n=N}^{\lfloor \tau_m \rfloor - L} \left|1 - \frac{z_m}{n+L+iI}\right| \left|1 - \frac{z_m}{-(n+L)+iI}\right|$$

$$\times \prod_{n=\lfloor \tau_m \rfloor - L+1}^{\lfloor \tau_m \rfloor + L} \left|1 - \frac{z_m}{t_n}\right| \left|1 - \frac{z_m}{-(n+L)+iI}\right|$$

$$\times \prod_{n=\lfloor \tau_m \rfloor + L+1}^{\infty} \left|1 - \frac{z_m}{n-L+iI}\right| \left|1 - \frac{z_m}{-(n+L)+iI}\right|$$

$$\geqslant C_2 \frac{\prod_{n=\lfloor \tau_m \rfloor - L+1}^{\lfloor \tau_m \rfloor + L} \left|1 - \frac{z_m}{t_n}\right|}{\prod_{n=N}^{N+L-1} \left|1 - \frac{z_m}{n+iI}\right| \left|1 - \frac{z_m}{-n+iI}\right|} \prod_{n=N}^{\infty} \left|1 - \frac{z_m}{n+iI}\right| \left|1 - \frac{z_m}{-n+iI}\right|.$$

The denominator is the modulus of a polynomial in z_m of degree $2L$. Thus, there exists a positive real number C_7 such that

$$|P(z_m)| \geqslant C_7 (R_m^+)^{-2L} \prod_{n=\lfloor \tau_m \rfloor - L+1}^{\lfloor \tau_m \rfloor + L} \left|1 - \frac{z_m}{t_n}\right| \prod_{n=N}^{\infty} \left|1 - \frac{z_m}{n+iI}\right| \left|1 - \frac{z_m}{-n+iI}\right| \tag{25}$$

for all $m \geqslant S_3$.

Let $n \geqslant \lfloor \tau_m \rfloor - L + 1$. Then

$$\sin |\theta_n| \leqslant \frac{I}{|t_n|} \leqslant \frac{I}{\Re(t_n)} \leqslant \frac{I}{n-L} \leqslant \frac{I}{\lfloor \tau_m \rfloor - 2L + 1} \leqslant \frac{I}{R_m^+ - 2I - 2L},$$

where we used (24) in the last step. On the other hand,

$$\sin \varphi \geqslant \frac{4I + 2L}{R_m^+}$$

and so

$$\frac{\sin \varphi}{\sin |\theta_n|} \geqslant \frac{4I+2L}{I} \left(1 - 2\frac{I+L}{R_m^+}\right) \geqslant \frac{4I+2L}{I} \left(1 - 2\frac{I+L}{4I+2L}\right) = 2.$$

This implies that

$$|\theta_n| \leqslant \min\{\varphi, \pi/6\}.$$

Along with a geometrical reflection, we arrive at

$$\begin{aligned}\left|1-\frac{z_m}{t_n}\right| &\geqslant \sin(\varphi-|\theta_n|)\\ &= \sin\varphi\left(\cos|\theta_n|-\cos\varphi\,\frac{\sin|\theta_n|}{\sin\varphi}\right)\\ &\geqslant \sin\varphi\left(\cos|\theta_n|-\frac{1}{2}\cos\varphi\right)\\ &\geqslant \frac{1}{2}\sin\varphi\cos|\theta_n|\\ &\geqslant \frac{\sqrt{3}}{4}\sin\varphi. \end{aligned}\tag{26}$$

Combining (20), (25), and (26) and applying Lemma 2.3, we obtain that

$$|G(z_m)| \geqslant C_8(R_m^+)^{-2L}\,|\sin\varphi|^{2L}\,|\sin(\pi(z_m-iI))| \tag{27}$$

for all $m \geqslant S_3$ and $C_8 := C_6 C_7(\sqrt{3}/4)^{2L}\, I/\sinh(\pi I)$.

Since

$$|\sin(x+iy)| \geqslant \frac{e^{|y|}-e^{-|y|}}{2} = \frac{e^{|y|}}{2}(1-e^{-2|y|}) \geqslant C_{10}e^{|y|} \tag{28}$$

for all $|y| \geqslant C_9 > 0$, where $C_{10} := (1-\exp(-2C_9))/2$, the inequality (18) follows from (27) in the case $\sin\varphi > (4I+2L)/R$.

Case 2. Let $0 \leqslant \Im(z_m) \leqslant 4I+2L$.

Defining again $p_n := \Re(t_n)$, we have

$$\left|1-\frac{z_m}{t_n}\right| \geqslant \left|1-\frac{x_m+iI}{p_n+iI}\right|$$

for all $n \geqslant N$. Analogous considerations show that

$$\left|1-\frac{z_m}{t_n}\right| \geqslant \left|1-\frac{x_m+iI}{n+L+iI}\right| \qquad \text{if} \quad n \leqslant x_m-L \tag{29}$$

and

$$\left|1-\frac{z_m}{t_n}\right| \geqslant \left|1-\frac{x_m+iI}{n-L+iI}\right| \qquad \text{if} \quad n \geqslant x_m+L. \tag{30}$$

Note that

$$\left|1+\frac{z_m}{n}\right| \geqslant \left|\frac{n+x_m}{n-iI}\right| = \left|1-\frac{x_m+iI}{-n+iI}\right|. \tag{31}$$

For all $m \geqslant S_3$ we have $\lfloor x_m \rfloor \geqslant N+L$. Therefore, after combining (23) and (29)–(31) we obtain that

$$\begin{aligned}
|P(z_m)| \geqslant C_2 &\prod_{n=N}^{\lfloor x_m \rfloor - L} \left|1-\frac{x_m+iI}{n+L+iI}\right| \left|1-\frac{x_m+iI}{-(n+L)+iI}\right| \\
&\times \prod_{n=\lfloor x_m \rfloor - L+1}^{\lfloor x_m \rfloor + L} \left|1-\frac{z_m}{t_n}\right| \left|1-\frac{x_m+iI}{-(n+L)+iI}\right| \\
&\times \prod_{n=\lfloor x_m \rfloor + L+1}^{\infty} \left|1-\frac{x_m+iI}{n-L+iI}\right| \left|1-\frac{x_m+iI}{-(n+L)+iI}\right| \\
= C_2 &\frac{\prod_{n=\lfloor x_m \rfloor - L+1}^{\lfloor x_m \rfloor + L} \frac{1}{|t_n|} |t_n - z_m|}{\prod_{n=N}^{N+L-1} \left|1-\frac{x_m+iI}{n+iI}\right| \left|1-\frac{x_m+iI}{-n+iI}\right|} \\
&\times \prod_{n=N}^{\infty} \left|1-\frac{x_m+iI}{n+iI}\right| \left|1-\frac{x_m+iI}{-n+iI}\right|
\end{aligned} \tag{32}$$

for all $m \geqslant S_3$. Note that

$$\lim_{m\to\infty} |R_m^+ - x_m| = 0. \tag{33}$$

Therefore, using the estimates (5) and (15), we can find a positive real number C_{11} such that

$$\prod_{n=\lfloor x_m \rfloor - L+1}^{\lfloor x_m \rfloor + L} \frac{1}{|t_n|} |t_n - z_m| \geqslant C_{11} (R_m^+)^{-2L}. \tag{34}$$

Since the denominator in (32) is the modulus of a polynomial in x_m of degree $2L$, there exists a positive real number C_{12} such that

$$\prod_{n=N}^{N+L-1} \left|1-\frac{x_m+iI}{n+iI}\right| \left|1-\frac{x_m+iI}{-n+iI}\right| \leqslant C_{12} (R_m^+)^{2L}. \tag{35}$$

Using (20) and (32)–(35) and applying Lemma 2.3 again, we arrive at

$$|G(z_m)| \geqslant C_{13}(R_m^+)^{-4L} \frac{|z_m - iI| \prod_{n=1}^{N-1} \left|1 - \frac{z_m}{n+iI}\right| \left|1 - \frac{z_m}{-n+iI}\right|}{x_m \prod_{n=1}^{N-1} \left|1 - \frac{x_m+iI}{n+iI}\right| \left|1 - \frac{x_m+iI}{-n+iI}\right|} |\sin(\pi x_m)| \tag{36}$$

for all $m \geqslant S_3$, where $C_{13} := C_2 C_6 C_{11} I/(C_{12} \sinh(\pi I))$.

Combining (14) and (33), we can choose an integer $S_4 \geqslant S_3$ so that $|x_m - n| > \eta/2$ for all $m \geqslant S_4$ and $n \in \mathbb{N}$. In particular, $|\sin(\pi x_m)|$ has the positive lower bound $\sin(\pi\eta/2)$. A simple discussion yields that the fraction in (36) has a positive lower bound C_{14}. Thus, we finally find that

$$|G(z_m)| \geqslant C_{15}(R_m^+)^{-4L},$$

where $C_{15} := C_{13} C_{14} \sin(\pi\eta/2)$. This completes the proof. ∎

Using the same techniques as in the proof of Lemma 2.4, we obtain

LEMMA 2.5. *Let $(t_n)_{n \in \mathbb{Z}}$ be a sequence of nodes satisfying* (1)–(5). *Let G be the canonical product corresponding to $(t_n)_{n \in \mathbb{Z}}$.*

Then there exists a positive real number $y_0 > I$ so that

$$|G(iy)| \geqslant C_{16}\, y^{-2L} e^{\pi(y-I)}, \tag{37}$$

$$|G(-iy)| \geqslant C_{17}\, y^{-2L} e^{\pi(y-I)} \tag{38}$$

for all $y \geqslant y_0$. The positive real numbers C_{16} and C_{17} are independent of y.

Proof. We shall indicate only the proof of (37). As above, we can find positive real numbers $y_0 > 2I$ and C_{18} such that

$$|G(iy)| \geqslant C_{18}\, |y - I| \prod_{n=1}^{N-1} \left|1 - \frac{iy}{n+iI}\right| \left|1 - \frac{iy}{-n+iI}\right|$$

$$\times \prod_{n=N}^{\infty} \left|1 - \frac{iy}{t_n}\right| \left|1 - \frac{iy}{-(n+L)+iI}\right|$$

for all $y \geqslant y_0$. Defining $p_n := \Re(t_n)$, we have

$$\left|1 - \frac{iy}{t_n}\right|^2 \geqslant \frac{p_n^2 + (y-I)^2}{p_n^2 + I^2}.$$

A discussion of the function

$$\tilde{f}(t) := \frac{t^2 + (y-I)^2}{t^2 + I^2}$$

yields that

$$\left|1 - \frac{iy}{t_n}\right| \geqslant \left|1 - \frac{iy}{n+L+iI}\right|$$

for all $n \geqslant N$ and $y > 2I$.

Applying Lemma 2.3 and the estimate (28), we can establish (37) by means of some simple calculations. ∎

3. PROOFS OF THE RESULTS

Proof of the Theorem. Since

$$\frac{G(z)}{(z-t_n)\, G'(t_n)} = \begin{cases} 1 & \text{if} \quad z = t_n \\ 0 & \text{if} \quad z = t_m \quad (m \neq n), \end{cases}$$

it suffices to prove (9) for $z \neq t_n$ $(n \in \mathbb{Z})$.

Now we consider the positively oriented Jordan curves $S_{m,n}$ defined by

$$\begin{aligned} S_{m,n} := {} & \{R_m^+ e^{i\varphi} : \varphi \in (-\pi/2, \pi/2)\} \cup [iR_m^+, iR_n^-] \\ & \cup \{R_n^- e^{i\varphi} : \varphi \in (\pi/2, 3\pi/2)\} \cup [-iR_n^-, -iR_m^+] \end{aligned}$$

for $m, n \geqslant N$ and the contour integral $I_{m,n}(z)$ defined by

$$I_{m,n}(z) := \frac{1}{2\pi i} \int_{S_{m,n}} \frac{f(\zeta)\, \Phi(\zeta - z)}{(\zeta - z)\, G(\zeta)}\, d\zeta$$

for $m, n \geqslant S$ and $z \in \mathbb{C} \backslash S_{m,n}$ ($S \in \mathbb{N}$ chosen according to Lemma 2.1). Let m and n in the following be large enough for z to lie in the interior of the Jordan curves $S_{m,n}$. Then using the residue theorem, we find that

$$I_{m,n}(z) = \frac{f(z)\, \Phi(0)}{G(z)} + \sum_{i=-n}^{m} \frac{f(t_i)\, \Phi(t_i - z)}{(t_i - z)\, G'(t_i)}$$

and so

$$f(z)\, \Phi(0) = I_{m,n}(z)\, G(z) + \sum_{i=-n}^{m} f(t_i)\, \frac{\Phi(t_i - z)}{z - t_i}\, \frac{G(z)}{G'(t_i)}.$$

Therefore, to prove (9) we must only show that

$$\lim_{m\to\infty}\ \lim_{n\to\infty} I_{m,n}(z)=0$$

for all complex numbers z which are different from t_k ($k\in\mathbb{Z}$) with uniform convergence if z lies in a compact subset of $\mathbb{C}$.

Using the assumptions (7) and (8), we may apply a well-known estimate for entire functions of exponential type [3, Lemma 2; 5, Lemmas 1 and 2] to obtain that

$$|f(Re^{i\varphi})\,\Phi(Re^{i\varphi}-z)|\leqslant C_1(z)\,R^{-4L}e^{\pi R\,|\sin\varphi|}$$

for all positive real numbers R, $\varphi\in[0,2\pi]$ and $z\in\mathbb{C}$.

Let $|z|\leqslant M$ for a positive real number M. Without loss of generality we may assume that $R_m^+\leqslant R_n^-$.

Then, applying Lemmas 2.4 and 2.5, we find for all $m, n\geqslant \max\{M+4I+3L+1, S, y_0+L\}$ (S and y_0 chosen according to Lemmas 2.4 and 2.5) that

$$\begin{aligned}
&2\pi\,|I_{m,n}(z)|\\
&\leqslant\int_{-\pi/2}^{\pi/2}\left|\frac{f(R_m^+e^{i\varphi})\,\Phi(R_m^+e^{i\varphi}-z)}{(R_m^+e^{i\varphi}-z)\,G(R_m^+e^{i\varphi})}R_m^+\right|d\varphi+\int_{R_m^+}^{R_n^-}\left|\frac{f(iy)\,\Phi(iy-z)}{(iy-z)\,G(iy)}\right|dy\\
&\quad+\int_{\pi/2}^{3\pi/2}\left|\frac{f(R_n^-e^{i\varphi})\,\Phi(R_n^-e^{i\varphi}-z)}{(R_n^-e^{i\varphi}-z)\,G(R_n^-e^{i\varphi})}R_n^-\right|d\varphi+\int_{-R_n^-}^{-R_m^+}\left|\frac{f(iy)\,\Phi(iy-z)}{(iy-z)\,G(iy)}\right|dy\\
&\leqslant\frac{2C_1(z)}{C_4}\frac{R_m^+}{R_m^+-M}\int_0^{\arcsin((4I+2L)/R_m^+)}e^{\pi R_m^+\sin\varphi}\,d\varphi\\
&\quad+\frac{2C_1(z)}{C_4}\frac{R_m^+}{R_m^+-M}\int_{\arcsin((4I+2L)/R_m^+)}^{\pi/2}\frac{e^{\pi R_m^+\sin\varphi}}{(R_m^+)^{2L}\,(\sin\varphi)^{2L}\,e^{\pi(R_m^+\sin\varphi-I)}}\,d\varphi\\
&\quad+\frac{2C_1(z)}{C_5}\frac{R_n^-}{R_n^--M}\int_0^{\arcsin((4I+2L)/R_n^-)}e^{\pi R_n^-\sin\varphi}\,d\varphi\\
&\quad+\frac{2C_1(z)}{C_5}\frac{R_n^-}{R_n^--M}\int_{\arcsin((4I+2L)/R_n^-)}^{\pi/2}\frac{e^{\pi R_n^-\sin\varphi}}{(R_n^-)^{2L}\,(\sin\varphi)^{2L}\,e^{\pi(R_n^-\sin\varphi-I)}}\,d\varphi\\
&\quad+2C_1(z)\max\left\{\frac{1}{C_{16}},\frac{1}{C_{17}}\right\}\frac{1}{R_m^+-M}\int_{R_m^+}^{R_n^-}\frac{e^{\pi y}}{y^{2L}e^{\pi(y-I)}}\,dy.
\end{aligned}$$

Using the inequalities

$$\frac{2}{\pi} x \leqslant \sin x \leqslant x \qquad \text{for} \quad 0 \leqslant x \leqslant \frac{\pi}{2}$$

and

$$y \leqslant \arcsin y \leqslant \frac{\pi}{2} y \qquad \text{for} \quad 0 \leqslant y \leqslant 1$$

to simplify the integrals, we obtain that

$$\begin{aligned} 2\pi\, |I_{m,n}(z)| \leqslant{}& C_{19} \int_0^{(\pi/2)((4I+2L)/R_m^+)} e^{\pi R_m^+ x}\, dx \\ &+ C_{19} \frac{e^{\pi I}}{(R_m^+)^{2L}} \int_{((4I+2L)/R_m^+)}^{\pi/2} \left(\frac{2}{\pi} x\right)^{-2L} dx \\ &+ C_{19} \int_0^{(\pi/2)((4I+2L)/R_n^-)} e^{\pi R_n^- x}\, dx \\ &+ C_{19} \frac{e^{\pi I}}{(R_n^-)^{2L}} \int_{((4I+2L)/R_n^-)}^{\pi/2} \left(\frac{2}{\pi} x\right)^{-2L} dx \\ &+ C_{20} \frac{e^{\pi I}}{R_m^+ - M} \int_{R_m^+}^{\infty} x^{-2L}\, dx, \end{aligned}$$

where

$$C_{19} := 2 \sup\{ C_1(\zeta) \colon |\zeta| \leqslant M\} \max\left\{\frac{1}{C_4}, \frac{1}{C_5}\right\} \frac{M+4I+2L+1}{4I+2L+1} < \infty$$

and

$$C_{20} := 2 \sup\{ C_1(\zeta) \colon |\zeta| \leqslant M\} \max\left\{\frac{1}{C_{16}}, \frac{1}{C_{17}}\right\} < \infty.$$

After some simple calculations we finally find that

$$|I_{m,n}(z)| \leqslant C_{21} \max\left\{\frac{1}{R_m^+}, \frac{1}{R_n^-}\right\}$$

for a positive real number C_{21} which is independent of z. This completes the proof. ▮

Proof of Corollary 1.2. We choose $\Phi = \psi((1+\alpha)/2, \varepsilon, \cdot)$ in the theorem. Therefore, we must only prove that (8) is valid with $\sup\{ C_1(\zeta) \colon |\zeta| \leqslant T\} < \infty$ for all positive real numbers T.

For all $y_0 > 0$ we have

$$|\Phi(x+iy)| = O\left(\exp\left(-\frac{|x|}{(\log|x|)^{\alpha}}\right)\right) \qquad \text{as} \quad x \to \pm\infty$$

uniformly for $|y| \leqslant y_0$ [13, Lemma 1].

Since

$$\lim_{x\to\infty} x^{4L} \exp\left(-x\left(\frac{1}{(\log x)^{\alpha_1}} - \frac{1}{(\log x)^{\alpha_2}}\right)\right) = 0$$

for all $1 < \alpha_1 < \alpha_2$, the corollary is established. ∎

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